

How (Not) to Palatini

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ABSTRACT

We revisit the problem of defining non-minimal gravity in the first order formalism. Specializing to scalar-tensor theories, which may be disguised as ‘higher-derivative’ models with the gravitational Lagrangians that depend only on the Ricci scalar, we show how to recast these theories as Palatini-like gravities. The correct formulation utilizes the Lagrange multiplier method, which preserves the canonical structure of the theory, and yields the conventional metric scalar-tensor gravity. We explain the discrepancies between the naïve Palatini and the Lagrange multiplier approach, showing that the naïve Palatini approach really swaps the theory for another. The differences disappear only in the limit of ordinary General Relativity, where an accidental redundancy ensures that the naïve Palatini works there. We outline the correct decoupling limits and the strong coupling regimes. As a corollary we find that the so-called ‘Modified Source Gravity’ models suffer from strong coupling problems at very low scales, and hence cannot be a realistic approximation of our universe. We also comment on a method to decouple the extra scalar using the chameleon mechanism.

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1 Introduction

In gravity the separation of matter and geometry may be somewhat arbitrary. Already Einstein pondered the issue of separating the matter stress energy from geometry, especially when facing the issue of cosmological constant. Thus, given that we don't know yet what is the underlying fundamental theory of the world, if one such exists, it is amusing to consider possibilities where some phenomena usually attributed to gravitating matter could really be of geometric origin. This is more so when we face the problem of explaining the dark sector of the universe, comprising over 95% of the current cosmic inventory. While there are well motivated particle physics models that can accommodate the dark matter part of the missing mass, there really are no universally appealing models of dark energy. It therefore seems reasonable to ask if the observed cosmic acceleration might not really be due to some failure of gravity to obey Einstein's General Relativity at the largest scales, instead of the presence of some repulsive medium filling up the world.

Among the simplest attempts to describe deviations from General Relativity, which attracted a good deal of attention recently, are the so-called $f(R)$ theories [1]-[4]. However, in their simplest variant, where one declares that R is the conventional Ricci scalar built out of the metric and its derivatives, these models are nothing but general scalar-tensor theories [5, 6] in disguise [7]-[9]. In fact, the history of such models is long and well known, as exemplified by a number of works [10]-[15] which analyzed the perturbative spectra of higher derivative gravity. These works showed that as long as the higher derivative terms only appear as powers of the Ricci scalar, one can always go to a unitary gauge, decoupling the one extra scalar degree of freedom from the helicity-2 mode in the metric¹. A simple demonstration, originally due to [12] and resurrected recently in [8], goes as follows. Starting with

$$S = \int d^4x \sqrt{-g} f(R), \quad (1)$$

one can introduce an auxiliary scalar field φ to rewrite the gravitational action (1) in a simpler way, as $S = \int d^4x \sqrt{-g} (f(\varphi) + \partial_\varphi f(\varphi)(R - \varphi))$. Then defining a new variable $\Phi = \partial_\varphi f(\varphi)$, and inverting this equation to find $\varphi(\Phi)$ yields

$$S = \int d^4x \sqrt{-g} \left(\Phi R - V(\Phi) \right), \quad (2)$$

with $V(\Phi) = \varphi \partial_\varphi f(\varphi)|_{\varphi=\varphi(\Phi)} - f(\varphi)$, which indeed is a special case of a general scalar-tensor theory [6] given by $S_{\text{BD}} = \int d^4x \sqrt{-g} \left(\Phi R - \frac{w}{\Phi} (\nabla \Phi)^2 - V(\Phi) \right)$, with $w = 0$. Thus it appears that resorting to $f(R)$ Lagrangians really brings nothing new or original, instead simply being a special case of the well known and extensively studied scalar-tensor gravity. Actually, to be a little more accurate, we should note that the map between the $f(R)$ theory and its scalar-tensor avatar is in fact a Legendre transform². This implies that the scalar-

¹In more complicated derivative expansions, one will in general meet monsters: ghosts and tachyons. If such are avoided, one may still have vector excitations [10]. However when Lorentz symmetry is unbroken these only couple to matter via higher-order terms which automatically suppresses their long range forces.

²Which seems to naturally arise in the context of dark energy, see the relevant discussion in the introduction of [16].

tensor theory (2) really describes a one parameter family of $f(R)$ models, which are obtained by Legendre-inverting (2) and are counted by a free integration constant.

A different twist came from some considerations of the Palatini, or the first order, formulation of $f(R)$ models [3],[17]-[23]. It was claimed that the Palatini formulation of such models is intrinsically *different* from the standard metric formulation. In fact it should be pointed out that this apparent discrepancy between the Palatini and the metric formulations of scalar-tensor gravities had already been noticed in [24]-[27]. It then seemed that in the Palatini approach in some cases the extra scalar field may altogether disappear from the spectrum of propagating modes, leaving in its wake only an algebraic constraint. If true, this would have been interesting, since it might have provided a different avenue for modifying General Relativity while respecting Solar System tests of gravity. This issue has been discussed extensively in [3],[17]-[29], whose authors seem to have accepted the view that the same action may yield different dynamics when formulated in terms of the metric alone or in terms of the connexion and the metric. This seems very puzzling, and begs the question about the nature of the variational procedure and the canonical structure of such formulations. On the other hand opinions on whether such formulations are physically reasonable or not differ. In particular, a special case where the scalar seemed to be non-dynamical, instead of merely inducing a matter sector constraint, has been advertised by [21], under the label of ‘Modified Source Gravity’. These authors hoped that large scale dynamics of such models may be reliably approximated by FRW universes and went on to explore it as an alternative cosmological theory. However in [18] Flanagan has already showed that in such theories electrons interact differently than in the usual electrodynamics, thus conflicting with what we see in nature. Nonetheless, the debate about such models and various hybrids [19, 22, 23], and their physical viability, seems to be ongoing.

In this note we revisit the Palatini formulation of nonminimal gravities and clarify how to define such theories. In fact, the correct first order formulation of a general gravity theory can be obtained by paralleling the first order formulation of gauge theories, in particular electromagnetism. This route has already been delineated in the classic work by C. Lanczos some fifty years ago [30]. We will work explicitly with scalar-tensor theories, because this automatically covers all the $f(R)$ models as well. The reason is that the trick explained above [12, 8] that relates Eqs. (1) and (2) remains perfectly applicable in any generally covariant theory where R is a scalar, independent of how it depends on the metric, connexion et cetera. Utilizing the Lagrange multiplier method, to safeguard the canonical structure of the theory we will establish clearly the correspondence between the Palatini and the metric formulation of scalar-tensor gravities, unveiling the limitations of the commonly used naïve Palatini approach. Similar approach has been applied recently in $2+1$ gravity by S. Deser [31], with conclusions similar to ours³ will see that the discrepancies between the naïve Palatini and the standard formulations of nonminimal gravity can be attributed to the missing terms in the canonical momentum of the scalar field variable, which shifts the scale of strong coupling of the theory. These differences disappear in the limit of ordinary General Relativity, where the scalar field completely decouples from the matter sector. In the cases without scalar self-interactions, its perturbative couplings to matter are protected by a shift symmetry

³We are grateful to S. Deser for bringing [31] to our attention.

$\phi \rightarrow \phi + \mathcal{C}$. In this limit, there also arises an accidental redundancy which decouples the Lagrange multiplier field and ensures that the naïve Palatini and the properly defined theory coincide as $w \rightarrow \infty$. We will also outline the correct strong coupling regimes and contrast them with the decoupling limits. As a corollary we will find that the so-called ‘Modified Source Gravity’ models suffer from strong coupling problems at very low scales, and not merely additional couplings which were discussed in [18]. Hence such models cannot be a realistic approximation of our universe. We will however point out one open avenue to help suppress the scalar mode, based on the chameleon mechanism.

The paper is organized as follows. In the next section, we will present a mechanical toy model which serves as a simple yet complete arena to illustrate the canonical properties of scalar tensor gravity in first and second order formulations. We will use it to illustrate the origin of the discrepancies between the naïve Palatini approach and the correct formulation as well as to study the symmetries which emerge in the decoupling limit. In section 3 we will show that a general scalar-tensor gravity shares precisely the same canonical properties and has identical accidental symmetries as our mechanical example. Section 4 is devoted to the discussion of decoupling versus strong coupling, and physical interpretation of the results. We will also explain the strong coupling problem which invalidates the ‘Modified Source Gravity’ there, and comment on the possibility of using chameleons. We summarize in section 5.

Before proceeding, we need to stress one more important point. In a significant subset of the existing literature on scalar-tensor gravities there is still an ongoing debate about which conformal frame of the theory is ‘physical’, implying that different frames lead to different physical answers. This debate is nugatory. Physical observables are frame independent, once the complete low-energy theory of gravity and matter is correctly specified, including the UV regulator, which may be modelled merely as a fixed cutoff scale. Hence we will freely hop between different conformal frames as our convenience dictates, knowing full well that the physical conclusions do not change. We will not revel in a more detailed justification of this fact, merely directing an interested reader (and perhaps an occasionally doubting one) to the classic treatise at the source of the scalar-tensor theories [5], whose clarity and lucidity we cannot hope to surpass. We will however stipulate precisely how the variables change in going from one frame to another.

2 A Mechanical Toy Model

To immediately zero in on the dynamical issues and step over the complications with indices, we begin our discussion with an example from particle mechanics. It faithfully represents the general problem with first order formulation of scalar-tensor gravity. Let us define a canonical system with an action

$$S = \int dt \left(\frac{w}{2Q} x \dot{Q}^2 - x Q \dot{Y} - \frac{1}{2} x Q Y^2 - x J \right), \quad (3)$$

where x is the analogue of $\sqrt{g}g^{\mu\nu}$, Y stands for $\Gamma_{\nu\lambda}^\mu$, and Q for the Brans-Dicke field Φ . The term xJ symbolizes the minimal couplings to matter sources J . With field redefinitions

$$z = xQ, \quad Q = e^{\frac{q}{\sqrt{w}}}, \quad (4)$$

we can rewrite this action as an analogue of Brans-Dicke theory in the Einstein frame,

$$S = \int dt \left(\frac{z}{2} \dot{q}^2 - z\dot{Y} - \frac{z}{2} Y^2 - ze^{-\frac{q}{\sqrt{w}}} J \right). \quad (5)$$

Using the standard variational technique to get the equations of motion $\frac{\dot{q}^2}{2} = \dot{Y} + \frac{Y^2}{2} + e^{-\frac{q}{\sqrt{w}}} J$, $(z\dot{q})' = \frac{z}{\sqrt{w}} e^{-\frac{q}{\sqrt{w}}} J$ and $Y = \frac{\dot{z}}{z}$, we can eliminate Y from the last equation, to finally obtain

$$\frac{\dot{q}^2}{2} = \left(\frac{\dot{z}}{z} \right)' + \frac{\dot{z}^2}{2z^2} + e^{-\frac{q}{\sqrt{w}}} J, \quad (z\dot{q})' = \frac{z}{\sqrt{w}} e^{-\frac{q}{\sqrt{w}}} J. \quad (6)$$

These equations are the same as what one would get by varying the action (5) *after* substituting $Y = \frac{\dot{z}}{z}$ directly into it:

$$S = \int dt \left(\frac{z}{2} \dot{q}^2 + \frac{\dot{z}^2}{2z} - ze^{-\frac{q}{\sqrt{w}}} J \right). \quad (7)$$

This is analogous to the naïve Palatini formulation of General Relativity with an extra scalar field, which however happens to have non-minimal couplings to matter J , parameterized by the coupling $1/\sqrt{w}$ after the transformation to the Einstein frame.

The confusion arises if we try to recast our theory in the original Brans-Dicke frame in the first order language. The relations which normally follow in the metric formulation imply $Y = \frac{\dot{x}}{x}$, and setting this in the action yields $S = \int dt \left(\frac{w}{2Q} x \dot{Q}^2 - xQ \left(\frac{\dot{x}}{x} \right)' - \frac{Q}{2} \frac{\dot{x}^2}{x} - xJ \right)$. Applying the field redefinitions (4) *now*, to this action, yields

$$S = \int dt \left(\left(1 - \frac{1}{w} \right) \frac{z}{2} \dot{q}^2 + \frac{\dot{z}^2}{2z} - ze^{-\frac{q}{\sqrt{w}}} J \right), \quad (8)$$

which clearly differs from Eq. (7) by the presence of the piece $\propto \frac{1}{w}$ in the first term. This only disappear in the limit $w \rightarrow \infty$, when the scalar decouples from the matter sources J , with decoupling here being tantamount to requiring that q is bounded. This difference is *precisely* the ‘ambiguity’ which has been noticed in the attempts to define the Palatini form of scalar-tensor gravity [24]-[27], [3, 18, 19].

What’s going on? Recall that in the naïve implementation of the first order formalism in a nonminimal theory there may be ambiguities in how to relate the connexion to the field variables. To resolve these ambiguities, in a given theory we should enforce the relation between the connexion and the field derivatives, or momenta, as a constraint, using a Lagrange multiplier, as prescribed in [30]. This means that our example above, which yields (8) really corresponds to picking the initial action

$$S = \int dt \left(\frac{w}{2Q} x \dot{Q}^2 - xQ \dot{Y} - \frac{xQ}{2} Y^2 - xJ + \lambda \left(Y - \frac{\dot{x}}{x} \right) \right). \quad (9)$$

Now we can compare this to the naïve Palatini formulation without the Lagrange multiplier, whose dynamics is encoded in (7). Applying the field redefinition (4), we can rewrite the action (9) as

$$S = \int dt \left(\frac{z}{2} \dot{q}^2 - z \dot{Y} - \frac{z}{2} Y^2 - z e^{-\frac{q}{\sqrt{w}}} J + \lambda \left(Y - \frac{\dot{z}}{z} + \frac{\dot{q}}{\sqrt{w}} \right) \right). \quad (10)$$

The straightforward variation of this action shows that the independent equations of motion reduce to (6) only when $w \rightarrow \infty$, corresponding to the decoupling limit of q . Indeed, we have seen that the equations (6) follow from the substitution of $Y = \frac{\dot{z}}{z}$ which arises as the equation of motion from action (5). We could therefore enforce this equation by a Lagrange multiplier at no cost, which would map (5) into the form (10), but in this case *without* the very last term $\propto \frac{\dot{q}}{\sqrt{w}}$ in the constraint. This term on the other hand vanishes when $w \rightarrow \infty$, rendering the two approaches coincident and simultaneously decoupling q from the matter sector J .

Note that in this limit the mode q becomes source-free, but it does *not* disappear from the equations of motion. Indeed, it still sources the mode z , and therefore it ‘gravitates’ as a free field. This is in fact completely consistent with the situation in scalar-tensor gravity. In the limit $w \rightarrow \infty$ the Brans-Dicke field does not completely drop out from the gravitational stress-energy tensor. Instead, its matter-induced sources $\propto T^\mu{}_\mu$ vanish since they are weighed by $1/\sqrt{w}$, but its own stress energy tensor transmutes to that of a matter-free field $\phi \sim \ln \Phi$ minimally coupled to gravity, just like q here. We will confirm this below, when we move beyond our mechanical analogy. We only note here that this provides an avenue for understanding the weakness of q - J couplings in a natural way, since it is protected by the shift symmetry $q \rightarrow q + \mathcal{C}$ which arises when $w \rightarrow \infty$. Another key question which begs one’s attention following the discussion above is, if the equations of motion which one finds following two *different* approaches to defining the canonical momenta of \dot{z} (which is of course what the Lagrange multiplier is enforcing) are different in general, why do they degenerate in the decoupling limit? While this is obvious from the equations of motion, it points to yet another emergent symmetry in the decoupling limit.

We now prove that this is precisely what happens, and illustrate the decoupling limit en route to our result. Let us first define

$$y = Y + \frac{\dot{q}}{\sqrt{w}}, \quad (11)$$

which allows us to rewrite the action (9) as

$$S = \int dt \left(z \left(y - \frac{\dot{z}}{z} \right) \frac{\dot{q}}{\sqrt{w}} + \left(z \frac{\dot{q}}{\sqrt{w}} \right)' + \left(1 - \frac{1}{w} \right) \frac{z}{2} \dot{q}^2 - z \dot{y} - \frac{z}{2} y^2 - z e^{-\frac{q}{\sqrt{w}}} J + \lambda \left(y - \frac{\dot{z}}{z} \right) \right). \quad (12)$$

Clearly, in the limit $w \rightarrow \infty$ where we hold z, y, q finite, the original variable Q related to q by Eq. (4) converges to a constant, $Q \rightarrow 1 + \mathcal{O}(\frac{q}{\sqrt{w}})$. Further, the variable q becomes source-free, as its mixing with J vanishes in this limit. Likewise, the first two terms in (12) also drop out, we find that $y \rightarrow Y$, and the resulting theory de facto coincides with (5) in

the limit $w \rightarrow \infty$. Conversely, for any finite value of w , y differs from Y as prescribed in (11), and the two theories are manifestly different from each other.

Further, let us now show that in the decoupling limit the variational equations that follow from (12) become degenerate, and the constraint equation, found by varying (12) with respect to λ becomes redundant. This can be seen directly from varying the full action (12) and noting that the variational equations force $\lambda = 0$ in the limit $w \rightarrow \infty$. So one may want to set $\lambda = 0$ directly into the action in this limit, amounting to the naïve Palatini approach. Yet, when this is consistent, one expects to have a symmetry which enforces the triviality of the constraint and decouples the Lagrange multiplier. To uncover this symmetry, let us consider the following deformations of the connexion variable y and the Lagrange multiplier λ , parameterized by a free parameter a :

$$y \rightarrow y + a, \quad \lambda \rightarrow \lambda + az. \quad (13)$$

Under this shift the action (12) changes to

$$\begin{aligned} S = & \int dt \left(z \left(y - \frac{\dot{z}}{z} \right) \frac{\dot{q}}{\sqrt{w}} + \left(z \frac{\dot{q}}{\sqrt{w}} \right)' + \left(1 - \frac{1}{w} \right) \frac{z}{2} \dot{q}^2 - z \dot{y} - \frac{z}{2} y^2 - z e^{-\frac{q}{\sqrt{w}}} J + \lambda \left(y - \frac{\dot{z}}{z} \right) \right) \\ & + \int dt \left(az \frac{\dot{q}}{\sqrt{w}} + \left(\lambda + \frac{az}{2} \right) a \right), \end{aligned} \quad (14)$$

where the first line is precisely our starting action (12), and the second line encodes the variation induced by the shift parameter a . Obviously, if we now choose $az/2 = -\lambda$, the second term in the second line of (14) vanishes, and so if also $w \rightarrow \infty$, the action remains invariant under this transformation of variables!

This means that in the decoupling limit the Lagrange multiplier drops out also, since the constraint which it enforces becomes redundant with one of the equations of motion in the theory. To see this, we can consider one-half of the shift (13), where we only change $y \rightarrow y + a$. Now (12) changes to

$$\begin{aligned} S = & \int dt \left(z \left(y - \frac{\dot{z}}{z} \right) \frac{\dot{q}}{\sqrt{w}} + \left(z \frac{\dot{q}}{\sqrt{w}} \right)' + \left(1 - \frac{1}{w} \right) \frac{z}{2} \dot{q}^2 - z \dot{y} - \frac{z}{2} y^2 - z e^{-\frac{q}{\sqrt{w}}} J \right) \\ & + \int dt \left(az \frac{\dot{q}}{\sqrt{w}} + (\lambda - az) \left(y - \frac{\dot{z}}{z} \right) + \left(\lambda - \frac{az}{2} \right) a \right), \end{aligned} \quad (15)$$

where now the top line is the action (12) with $\lambda = 0$, and the second line contains the complete dependence of the action on the Lagrange multiplier and the shift a . If we now pick a such that $az = \lambda$, we can decouple the Lagrange multiplier from the connexion. Indeed, referring to the first line of (14) as S_0 , we then find

$$S = S_0 + \int dt \left(\lambda \frac{\dot{q}}{\sqrt{w}} + \frac{\lambda^2}{2z} \right), \quad (16)$$

where S_0 is totally independent of λ . Clearly when $w \rightarrow \infty$ and q is finite, λ completely decouples from the theory. Its dependence of the action in this limit reduces to

$$S = S_0 + \int dt \frac{\lambda^2}{2z}, \quad (17)$$

and we can integrate it out. Alternatively, since the action is quadratic and algebraic in λ , its variational equation in this gauge is simply $\lambda = 0$ independently of any other variables, and we can drop it from the theory. The constraint which λ was introduced to enforce will follow from the nontrivial variational equations that remain, as it must since it follows in the original gauge. Therefore, we see that in the decoupling limit, and only then, we can simply substitute the constraint $Y = \frac{\dot{z}}{z}$ in the action while maintaining the canonical structure of the theory because of the accidental symmetry.

3 Beyond Mechanics: a Scalar, Gravity, and Matter

We now turn to some generalized scalar-tensor gravity given by a generalization of the Brans-Dicke action [6]

$$S_{\text{BD}} = \int d^4x \sqrt{-g} \left(\Phi R - \frac{w}{\Phi} \nabla^\mu \Phi \nabla_\mu \Phi - V(\Phi) - \mathcal{L}_{\text{matter}}(g^{\mu\nu}, \Psi) \right). \quad (18)$$

This action, as we discussed above, includes all $f(R)$ models after suitable field redefinitions [12, 8, 18]. Here we work with the convention $R_{\mu\nu} = \partial_\rho \Gamma_{\mu\nu}^\rho - \dots$, represent the matter fields by Ψ , and assume, for simplicity, that they are minimally coupled to the Brans-Dicke frame metric $g_{\mu\nu}$. In this, we also take all the space-time covariant derivatives which may appear in the matter action to depend on the same connexion coefficients $\Gamma_{\nu\lambda}^\mu$ as the one which appear in the curvature R . Then, following the standard prescription for the first order formulation [30], explained in the mechanical example of the previous section, we add to the action (18) the Lagrange multiplier terms which enforce Brans-Dicke frame metric compatibility⁴,

$$S_{\text{LM}} = \int d^4x \sqrt{-g} \Phi \lambda_{\nu\rho}^\mu \nabla_\mu g^{\nu\rho}, \quad (19)$$

with the Lagrange multiplier tensor fields $\lambda_{\nu\rho}^\mu$ normalized to include the prefactor Φ for later convenience.

To go to the Einstein frame, we use the field redefinitions

$$\tilde{g}_{\mu\nu} = e^{\frac{\phi}{\sqrt{w}M_P}} g_{\mu\nu}, \quad \Phi = \frac{M_P^2}{2} e^{\frac{\phi}{\sqrt{w}M_P}}, \quad (20)$$

where we are including \sqrt{w} in the redefinition (20) in order to facilitate taking the decoupling limit $w \rightarrow \infty$ as in the mechanical example of the previous section. To compare our formulas with the familiar scalar-tensor gravities one can simply rescale the scalar field $\phi \rightarrow \sqrt{w}\phi$ at any desired step. Since we are working with the first order formalism, where the connexion is an independent variable as opposed to being given by the metric derivatives, the conformal transformation (20) does *not* automatically induce the shift of the connexion in passing from one frame to another. Instead, the shift, if any, must be derived from the equations of motion generated by varying the full action $S = S_{\text{BD}} + S_{\text{LM}}$. To see how, we first rewrite it

⁴One can choose to enforce metric compatibility in any other frame conformally related to Brans-Dicke by a function of Φ . This would yield a continuous family of scalar-tensor theories parameterized by $w(\Phi)$.

in terms of the new variables (20), holding the connexion and the Ricci tensor fixed. Using $R = g^{\mu\nu} R_{\mu\nu} = e^{\frac{\phi}{\sqrt{w}M_P}} \tilde{g}^{\mu\nu} R_{\mu\nu}$, and similarly $(\nabla\phi)^2 = e^{\frac{\phi}{\sqrt{w}M_P}} (\tilde{\nabla}\phi)^2$, we find

$$S = \int d^4x \sqrt{-\tilde{g}} \left(\frac{M_P^2}{2} \tilde{g}^{\mu\nu} R_{\mu\nu} - \frac{1}{2} (\tilde{\nabla}\phi)^2 + \frac{M_P^2}{2} \lambda_{\nu\rho}^\mu \left(\frac{1}{\sqrt{w}M_P} \tilde{g}^{\nu\rho} \partial_\mu \phi + \nabla_\mu \tilde{g}^{\nu\rho} \right) - \tilde{V}(\phi) - \tilde{\mathcal{L}}_{\text{matter}} \right), \quad (21)$$

where $\tilde{V}(\phi) = e^{-2\phi/\sqrt{w}M_P} V(\frac{M_P^2}{2} e^{\frac{\phi}{\sqrt{w}M_P}})$ and $\tilde{\mathcal{L}}_{\text{matter}} = e^{-2\phi/\sqrt{w}M_P} \mathcal{L}_{\text{matter}}(e^{\phi/\sqrt{w}M_P} \tilde{g}^{\mu\nu}, \Psi)$. Next, we redefine the connexion $\Gamma_{\nu\rho}^\mu$ according to $\Gamma_{\nu\rho}^\mu = \tilde{\Gamma}_{\nu\rho}^\mu + c^\mu{}_{\nu\rho}$, where

$$c^\mu{}_{\nu\rho} = -\frac{1}{2\sqrt{w}M_P} (2\delta_{(\nu}^\mu \partial_{\rho)} \phi - \tilde{g}^{\mu\lambda} \tilde{g}_{\nu\rho} \partial_\lambda \phi). \quad (22)$$

Then we extract the $c^\mu{}_{\nu\rho}$ -dependent terms out of $\tilde{g}^{\mu\nu} R_{\mu\nu}$ by defining $\tilde{R}_{\mu\nu} = \partial_\rho \tilde{\Gamma}_{\mu\nu}^\rho - \dots$ and $\tilde{R} = \tilde{g}^{\mu\nu} \tilde{R}_{\mu\nu}$, which allows us to rewrite the action (21) as

$$S = \int d^4x \sqrt{-\tilde{g}} \left(\frac{M_P^2}{2} \tilde{R} - \frac{1}{2} (\tilde{\nabla}\phi)^2 - \tilde{V}(\phi) - \tilde{\mathcal{L}}_{\text{matter}} + M_P^2 \tilde{g}^{\mu\nu} (\tilde{\nabla}_{[\rho} c^\rho{}_{\mu]\nu} + c^\rho{}_{\lambda[\rho} c^\lambda{}_{\mu]\nu}) + \frac{M_P^2}{2} \lambda_{\nu\rho}^\mu \tilde{\nabla}_\mu \tilde{g}^{\nu\rho} \right). \quad (23)$$

In this equation, we have used $\tilde{\nabla}$, the covariant derivative with respect to $\tilde{\Gamma}$, eliminating ∇ from the constraint in its favor by using Eq. (22).

This formulation of the theory in the first order formalism generates identical dynamics as the standard metric formulation. One can quickly verify this by first varying the action (21) with respect to the Lagrange multipliers, yielding the metric compatibility condition and expressing the connexion $\Gamma_{\nu\rho}^\mu$ in terms of the metric $\tilde{g}_{\mu\nu}$ and scalar field ϕ . Substituting this back in the action, which is completely algebraic with respect to the connexion apart from a boundary term, and varying the result with respect to the metric and the scalar field ϕ yields precisely the expected field equations of Brans-Dicke theory in the Einstein frame. Indeed, let us show this for the scalar case. Eliminating the connexion from the action, using the condition that it is the Christoffel symbol of $e^{-\phi/\sqrt{w}M_P} \tilde{g}_{\mu\nu}$, which yields

$$\tilde{g}^{\mu\nu} R_{\mu\nu} = \tilde{R} + \frac{3}{\sqrt{w}M_P} \tilde{\nabla}^2 \phi - \frac{3}{2wM_P^2} (\tilde{\nabla}\phi)^2, \quad (24)$$

where \tilde{R} is the standard Ricci scalar of $\tilde{g}_{\mu\nu}$, we can rewrite (21) as

$$S = \int d^4x \sqrt{-\tilde{g}} \left(\frac{M_P^2}{2} \tilde{R} - \frac{1}{2} \left(1 + \frac{3}{2w}\right) (\tilde{\nabla}\phi)^2 - \tilde{V}(\phi) - \tilde{\mathcal{L}}_{\text{matter}} \right). \quad (25)$$

Varying this with respect to ϕ yields $(1 + \frac{3}{2w}) \tilde{\nabla}^2 \phi - \partial_\phi \tilde{V} - \frac{\delta \tilde{\mathcal{L}}_{\text{matter}}}{\delta \phi} = 0$. Then we use the stress energy tensor in the Einstein frame based on the relation between $\tilde{\mathcal{L}}_{\text{matter}}$ and $\mathcal{L}_{\text{matter}}$,

$$\tilde{T}_{\mu\nu} = -\tilde{g}_{\mu\nu} e^{-\frac{2\phi}{\sqrt{w}M_P}} \mathcal{L}_{\text{matter}} + 2e^{-\frac{2\phi}{\sqrt{w}M_P}} \frac{\delta \mathcal{L}_{\text{matter}}}{\delta \tilde{g}^{\mu\nu}}, \quad (26)$$

and the chain rule to determine the derivatives of the Lagrangian, which immediately yields $\frac{\delta \tilde{\mathcal{L}}_{\text{matter}}}{\delta \phi} = \frac{\tilde{T}}{2\sqrt{w}M_P}$. Hence the scalar field equation is

$$\tilde{\nabla}^2 \phi = \frac{2w}{2w+3} \frac{\partial \tilde{V}}{\partial \phi} + \frac{\sqrt{w}}{(2w+3)M_P} \tilde{T}, \quad (27)$$

precisely as in the standard metric formulation. In particular, the limit $w \rightarrow \infty$ is the weak coupling limit, where the theory reduces to Einstein's General Relativity, with a scalar field with a potential $\tilde{V}(\phi)$, free of matter sources and minimally coupled to gravity⁵. If in addition the potential \tilde{V} vanishes, the theory is invariant under a shift symmetry $\phi \rightarrow \phi + \mathcal{C}$ which we noted above, and this provides an avenue for understanding weak scalar-matter couplings from the vantage point of naturalness. On the other hand, $w \rightarrow -\frac{3}{2}$ is the strong coupling limit, where even a tiniest self-interaction $\partial_\phi \tilde{V} \neq 0$ or a source $\tilde{T} \neq 0$ yield to a catastrophic response from the scalar field. If the self-interactions and the sources are completely absent, the scalar field does appear tame, behaving as a gravitating massless field, which can be checked by rescaling it as $\phi \rightarrow \phi/\sqrt{1 + \frac{3}{2w}}$. Hence in this limit the scalar behaves as a *poltergeist*: it needs a medium to summon it, upon which it backreacts violently. The presence of the medium is generic, however, which implies that the theory ceases to be under control.

We should contrast this with what happens in the more common naïve Palatini approaches [24]-[27], [17]-[23]. In the naïve Palatini formulation, the Lagrange multipliers and the constraints they enforce are missing. One gets the relation between the connexion and the metric by formally solving the algebraic equations for the connexion which one gets from varying (18). This picks out Einstein frame connexion as the solution, which one can readily verify. Hence the effective action for the metric and the scalar field is of the same form as (25), but *without* the terms $\propto \frac{3}{4w}(\tilde{\nabla}\phi)^2$. Clearly, this yields a different theory, which only coincides with the standard formulation in the decoupling limit $w \rightarrow \infty$, where the theory again reduces to General Relativity with a minimally coupled scalar field. The strong coupling limit now is significantly different than in the standard formulation, as can be seen from the scalar field equation,

$$\tilde{\nabla}^2 \phi = \frac{\partial \tilde{V}}{\partial \phi} + \frac{\tilde{T}}{2\sqrt{w}M_P}. \quad (28)$$

In this case, the strong coupling limit corresponds to $w \rightarrow 0$, where the direct matter-scalar couplings explode. If the sources vanish the field again tames up, behaving as a minimally coupled scalar (as long as its self-interactions are perturbative). Again, since nonvanishing sources are generic this outlines the end of the validity of the theory. We shall return to more discussion of strong coupling in the next section.

Next we explain why the difference between standard and the naïve Palatini formulation disappears in the decoupling limit $w \rightarrow \infty$. To this end, we exploit the tricks which we have

⁵We can check immediately that the coupling of ϕ to gravity does not disappear but reduces to minimal by writing the gravitational field equations, and substituting the limit $w \rightarrow \infty$. The surviving stress energy tensor from the scalar field ϕ acquires precisely the canonical form familiar from General Relativity.

learnt in the previous section, in dealing with our simple mechanical example, and again find an accidental symmetry in the decoupling limit which ensures that the two approaches coincide. We start by shifting the connexion to

$$\tilde{\Gamma}_{\nu\rho}^{\mu} \rightarrow \hat{\Gamma}_{\nu\rho}^{\mu} = \tilde{\Gamma}_{\nu\rho}^{\mu} + \lambda_{\nu\rho}^{\mu} - \frac{2}{3}\delta_{(\nu}^{\mu}\lambda_{\rho)}, \quad (29)$$

where $\lambda_{\rho} \equiv \lambda_{\lambda_{\rho}}^{\lambda}$ and parenthesis in the indices denote the symmetrization of the enclosed structures. This decouples the Lagrange multipliers $\lambda_{\nu\rho}^{\mu}$ from the connexion $\tilde{\Gamma}_{\nu\rho}^{\mu}$. Indeed, starting with the action (23) written for the hatted connexion symbols $\hat{\Gamma}_{\nu\rho}^{\mu}$ and substituting (29), we find

$$\begin{aligned} S = S_0 + \frac{M_P^2}{2} \int d^4x \sqrt{-\tilde{g}} & \left(2\tilde{g}^{\mu\nu} (\tilde{\nabla}_{[\rho} c^{\rho}_{\mu]\nu} + c^{\rho}_{\lambda[\rho} c^{\lambda}_{\mu]\nu}) \right. \\ & \left. + \tilde{g}^{\mu\nu} (\lambda_{\mu\lambda}^{\rho} \lambda_{\rho\nu}^{\lambda} - \frac{2}{3} \lambda_{\rho}^{\lambda} \lambda_{\mu\nu}^{\rho} - \frac{1}{3} \lambda_{\mu}^{\lambda} \lambda_{\nu}^{\rho}) + \tilde{g}^{\mu\nu} (\lambda_{\mu\nu}^{\rho} c^{\lambda}_{\lambda\rho} - 2\lambda_{\lambda(\mu}^{\rho} c^{\lambda}_{\nu)\rho}) \right), \end{aligned} \quad (30)$$

where $S_0 = \int d^4x \sqrt{-\tilde{g}} \left(\frac{M_P^2}{2} \tilde{R} - \frac{1}{2} (\tilde{\nabla}\phi)^2 - \tilde{V}(\phi) - \tilde{\mathcal{L}}_{\text{matter}} \right)$ is the standard Einstein-Hilbert action.

Now we can unveil the symmetry which explains why the standard formulation and the naïve Palatini approach coincide in the decoupling limit, as in the mechanical example above. Consider, in analogy with (13), the transformation

$$\tilde{\Gamma}_{\mu\nu}^{\sigma} \rightarrow \tilde{\Gamma}_{\mu\nu}^{\sigma} + a_{\mu\nu}^{\sigma}, \quad \lambda_{\mu\nu}^{\sigma} \rightarrow \lambda_{\mu\nu}^{\sigma} + a_{\mu\nu}^{\sigma} - \delta_{(\mu}^{\sigma} a_{\nu)}, \quad (31)$$

where $a_{\mu} = a_{\mu\nu}^{\nu}$, under which (23) transforms into

$$\begin{aligned} S \rightarrow S + \frac{M_P^2}{2} \int d^4x \sqrt{-\tilde{g}} & \left(\tilde{g}^{\mu\nu} (a_{\rho} c^{\rho}_{\mu\nu} + c_{\rho} a^{\rho}_{\mu\nu} - 2a^{\rho}_{\mu\sigma} c^{\sigma}_{\rho\nu}) \right. \\ & \left. + \tilde{g}^{\mu\nu} (2\lambda_{\mu\sigma}^{\rho} a^{\sigma}_{\rho\nu} + a^{\rho}_{\mu\sigma} a^{\sigma}_{\rho\nu} - a_{\mu} a_{\nu}) \right), \end{aligned} \quad (32)$$

up to a surface term that we neglect. In the decoupling limit $w \rightarrow \infty$, $c^{\sigma}_{\mu\nu}$ all vanish, as is clear from (22) and the fact that ϕ must remain bounded. Thus the action will be invariant under (31) if there exist fields $a_{\mu\nu}^{\sigma}$ such that the second line of (32) vanishes. It is straightforward to prove that this happens for

$$a^{\sigma}_{\mu\nu} = -2\lambda^{\sigma}_{\mu\nu} + \frac{10}{9}\delta^{\sigma}_{(\mu}\lambda_{\nu)} - \frac{2}{9}\delta^{\sigma}_{(\mu}\tilde{g}_{\nu)\rho}\tilde{g}^{\alpha\beta}\lambda^{\rho}_{\alpha\beta} + \frac{4}{9}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\lambda^{\sigma}_{\alpha\beta} - \frac{2}{9}\tilde{g}_{\mu\nu}\tilde{g}^{\sigma\alpha}\lambda_{\alpha}. \quad (33)$$

Therefore, in the $w \rightarrow \infty$ limit the theory enjoys the extra symmetry (31) with the transformation parameters related by (33). This points to the fact that the Lagrange multipliers are really irrelevant in the decoupling limit, since their value can be altered by the action of the symmetry transformation, without changing the action and the field equations. Indeed, if we return to the action (30), we see that when $w \rightarrow \infty$, $c \rightarrow 0$ it simply becomes

$$S = S_0 + \frac{M_P^2}{2} \int d^4x \sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} (\lambda_{\mu\lambda}^{\rho} \lambda_{\rho\nu}^{\lambda} - \frac{2}{3} \lambda_{\lambda}^{\rho} \lambda_{\mu\nu}^{\rho} - \frac{1}{3} \lambda_{\mu}^{\lambda} \lambda_{\nu}^{\rho}). \quad (34)$$

This action depends on $\lambda_{\nu\lambda}^\mu$ and its contractions only quadratically, such that its variation yields the equations $\lambda_{\nu\lambda}^\mu = 0$, allowing us to set the Lagrange multipliers directly to zero in the action. This leaves us precisely with the standard Einstein-Hilbert action of General Relativity coupled to matter and a scalar field,

$$S_0 = \int d^4x \sqrt{-\tilde{g}} \left(\frac{M_P^2}{2} \tilde{R} - \frac{1}{2} (\tilde{\nabla} \phi)^2 - \tilde{V}(\phi) - \tilde{\mathcal{L}}_{\text{matter}} \right). \quad (35)$$

This, of course, agrees with what we would get by taking $w \rightarrow \infty$ limit of the naïve Palatini formulation without the Lagrange multiplier. However, the agreement will only occur in this limit, showing what went wrong with the naïve approach: it swapped the theory for something else before the dynamics even began!

4 Strong Coupling versus Decoupling

To continue our discussion of the aspects of strong coupling, and contrast it with decoupling, we can rewrite the effective scalar tensor action in the Einstein frame using the variable

$$\chi = \frac{\phi}{\sqrt{w}}. \quad (36)$$

This extracts the w -dependence from the matter sector, and encodes it in the wave function renormalization $Z(w)$ of the scalar. So after integrating out the constraints, in these variables the action becomes

$$S = \int d^4x \sqrt{-\tilde{g}} \left(\frac{M_P^2}{2} \tilde{R} - \frac{Z(w)}{2} (\tilde{\nabla} \chi)^2 - \tilde{V}(\chi) - \tilde{\mathcal{L}}_{\text{matter}} \right), \quad (37)$$

where now $\tilde{V}(\chi) = e^{-2\chi/M_P} V(\frac{M_P^2}{2} e^{\frac{\chi}{M_P}})$ and $\tilde{\mathcal{L}}_{\text{matter}} = e^{-2\chi/M_P} \mathcal{L}_{\text{matter}}(e^{\chi/M_P} \tilde{g}^{\mu\nu}, \Psi)$, and

$$Z(w) = \begin{cases} w + \frac{3}{2} & \text{for the standard formulation,} \\ w & \text{for naïve Palatini.} \end{cases} \quad (38)$$

From here on we take it that all the w -dependence is fully coded in $Z(w)$, for simplicity. Our conclusions can be straightforwardly generalized to more complicated models.

With this in mind, it is now clear that the scalar sector runs into strong coupling as $Z(w) \rightarrow 0$. Indeed the field equation for χ which follows from (37) is

$$Z(w) \tilde{\nabla}^2 \chi = \partial_\chi \tilde{V} + \frac{\tilde{T}}{2M_P}, \quad (39)$$

which can be readily verified, for example, by substituting (36) and (38) into (27). Obviously, as $Z(w) \rightarrow 0$, the field χ cannot remain bounded in the presence of nontrivial sources or self-interactions. Things are however much worse quantum-mechanically. From the viewpoint of effective field theory, the action (37) should be understood as a two-derivative truncation of an infinite series, valid only when the terms left out are sufficiently small to

be neglected (a reader may consult [32] for more discussion and references). To check the circumstances under which this may be hoped for, we can use canonically normalized scalar $\hat{\chi} = \sqrt{Z(w)}\chi$, which implies that all of its polynomial self-couplings $\propto g^{(n)}\chi^n$ are given by $\hat{g}^{(n)} = g^{(n)}/Z^{n/2}(w)$, and its coupling to matter is set by $g = \frac{1}{\sqrt{Z(w)}M_P}$. They all diverge as $Z(w) \rightarrow 0$, implying that perturbation theory, which is the only *a priori* reason for the truncation of the dynamics to (37), breaks down. Thus the theory based on (37) alone is completely unreliable when $Z(w) \rightarrow 0$. This does not preclude a possibility that a consistent regulator exists, where (37) is replaced by something else, which is well behaved, restoring perturbativity of the framework. An example for this is the regularization of the 4-Fermi theory by the Standard Model. However, such a regulator must be found before one can attempt to extract any predictions from (37) in the strong coupling regime.

The action (37) also shows that our comparison of the scalar in strong coupling to a poltergeist is apt. If we had taken (37) seriously into, and beyond, strong coupling, and continue changing w past $Z(w) = 0$, the field would have become a ghost. This would turn on spontaneous instabilities in the scalar sector without the need for sources. Thus, in some sense the strong coupling regime is a welcome indicator of the impending deterioration of the spectrum into ghost-like modes. By itself it does not guarantee that the theory self-regulates, but merely indicates that we should look for a suitable completion, if such exists. However one turns this argument, it shows that the theory (37) must not be trusted when $Z(w) \rightarrow 0$ and beyond.

To illustrate this fact, let's consider what happens if we choose to turn the blind eye to the strong coupling warnings going up. We could just take the action (37), and the scalar field equation (39) and declare them valid when $Z(w) = 0$. This is exactly the idea behind the so-called ‘Modified Source Gravity’ proposal of [21]. Indeed, the motivation behind [21] is the following argument: take an $f(R)$ model, as in [1, 2], pick the scales to account for cosmic acceleration now, allow it to be mapped to a scalar-tensor theory, for which $w = 0$ [8], but define the dynamics using naïve Palatini approach. As we have seen above, this precisely selects (37) with the second line of (38), due to the fact that the naïve Palatini drops a term from the canonical momentum of the theory. Since $w = 0$ as in any $f(R)$ this means that the resulting effective theory is exactly in the strong coupling regime $Z(0) = 0$. Now, instead of throwing it away, [21] take the theory (37) and substitute $Z = 0$ directly in this action, choosing to trust it as such. Then the scalar field equation (39) degenerates into the constraint

$$\partial_\chi \tilde{V} + \frac{\tilde{T}}{2M_P} = 0, \quad (40)$$

and the field χ appears to be completely non-dynamical: a non-propagating mode, whose only job is as a Lagrange multiplier, to enforce (40). One could then take this equation, combine it with the resulting field equations governing gravity and matter, assume that some cosmological limit exists and go on comparing to conventional cosmology.

From the viewpoint of our previous discussion of the strong coupling limit which we reach by taking $Z \rightarrow 0$ continuously, however, the condition (40) is an extreme *fine tuning*. It requires constraining the matter sector so that its dynamics satisfies (40) with perfect precision in order to avoid sourcing $\tilde{\nabla}^2\chi$. Thus the matter fields must be subjected to

additional, abnormal dynamics, which differs from the parent matter theory. Flanagan has already observed this in [18], noting that in the naïve Palatini version of $f(R)$ gravity χ is an auxiliary field that enforces the constraint (40), and can be integrated out of the action inducing new operators in the matter sector. The example of [18] involved electrons, which pick up new derivative couplings, in conflict with the standard Maxwell theory.

However the situation is even more serious. Not only does the matter sector pick up new interactions, but its effective description altogether breaks down at a very low scale, much below the cutoff of $\sim \text{TeV}$, up to which the Standard Model should remain valid. In fact, when the scales of the $f(R)$ theory are picked to generate cosmic acceleration now, the matter sector becomes strongly coupled at about the millimeter scale! This shows that ‘Modified Source Gravity’ is meaningless at all scales below the millimeter, in complete conflict with what we have so far learned about Nature.

Let us illustrate this by using a scalar matter field, e.g. the Higgs field that controls the Standard Model masses, as a probe. Its flat space Lagrangian, to quadratic order in fields, is $\mathcal{L}_{\text{Higgs}} = \frac{1}{2}(\partial\mathcal{H})^2 + \frac{m_{\mathcal{H}}^2}{2}\mathcal{H}^2$. In this case Eq. (40) becomes

$$M_P \partial_\chi \tilde{V} = \frac{1}{2} e^{-\chi/M_P} (\tilde{\nabla}\mathcal{H})^2 + e^{-2\chi/M_P} m_{\mathcal{H}}^2 \mathcal{H}^2. \quad (41)$$

In any generic example, where we assume that the matter theory is perturbative so that the right-hand side of (41) is small, we can solve for χ by Taylor expanding about the vacuum value χ_* , and determining terms order-by-order [18]. For the model $f(R) = \frac{M_P^2}{2}(R - \frac{\mu^4}{R})$ studied by [1, 2], the scalar dual in the naïve Palatini approach is the theory (37) with $Z = 0$ and $\tilde{V}(\chi) = M_P^2 \mu^2 e^{-2\chi/M_P} (e^{\chi/M_P} - 1)^{1/2}$ [8, 18]. It has a vacuum that fits cosmic acceleration now if $\chi_* \simeq M_P$ and $\mu \simeq H_0 \sim 10^{-33} \text{eV}$. Around the vacuum χ_* we can approximate any such potential by $\tilde{V} = \Lambda^4 (1 + a \frac{\chi}{M_P} + \frac{b}{2} (\frac{\chi}{M_P})^2 + \dots)$ where $\Lambda^4 = 3\Omega_\Lambda M_P^2 H_0^2$ is the dark energy scale now, and a and b are $\mathcal{O}(1)$ numbers. The condition (41) then yields

$$\Lambda^4 \left(b \frac{\chi}{M_P} + a \right) = \frac{1}{2} e^{-\chi/M_P} (\tilde{\nabla}\mathcal{H})^2 + e^{-2\chi/M_P} m_{\mathcal{H}}^2 \mathcal{H}^2 + \dots \quad (42)$$

Solving for χ and substituting the result in $\tilde{V} + \tilde{\mathcal{L}}_{\text{Higgs}}$ produces the effective action, $\tilde{\mathcal{L}}_{\text{Higgs}}^{(\text{eff})}$, for \mathcal{H} . To illustrate our point, it suffices to only determine the first few derivative terms in the expansion, which are

$$\tilde{\mathcal{L}}_{\text{Higgs}}^{(\text{eff})} = \frac{1}{2} e^{-\chi_*/M_P} (\tilde{\nabla}\mathcal{H})^2 \left(\mathcal{O}(1) + \mathcal{O}(1) \times e^{-\chi_*/M_P} \frac{(\tilde{\nabla}\mathcal{H})^2}{\Lambda^4} + \dots \right). \quad (43)$$

Clearly, the subleading derivative terms become dominant at the momentum scale $k_S \sim \Lambda$. Thus at momenta $k > k_S$, or alternatively at distances $\ell < 1/k_S$ the matter sector theory by itself ceases to be valid, demanding a new UV completion. If $\Lambda \sim 10^{-3} \text{eV}$, this means that the Standard Model (not just the Higgs, but everything that depends on it) becomes nonperturbative at macroscopic scales. Thus, even if we attempt to ignore the strong coupling of the scalar χ and treat its equation at $Z = 0$ as a constraint (40), the strong coupling problem reappears in the effective description of the matter sector, lowering

the cutoff to whatever is the χ vacuum energy scale. If it is chosen to explain the cosmic acceleration now, the strong coupling occurs at a very low scale, $\Lambda \sim 10^{-3} \text{ eV}$. At this scale we must bring χ back into the theory, and subsequently face its strongly coupled dynamics. Hence one cannot really hide from strong coupling problem in ‘Modified Source Gravity’.

So far all of our dynamical analysis has been done around the vacuum of the theory. If one applies observational bounds to it, one then finds that models which involve gravitationally, or more strongly, coupled, light fields that give rise to dynamical dark energy are excluded [33] or worse yet, do not even make sense. However, an avenue for hiding such a field still remains open, albeit only narrowly. It concerns decoupling the extra degrees of freedom with environmental effects, using the chameleon mechanism [34]. We ought to emphasise that although this mechanism cannot rescue ‘Modified Source Gravity’ from strong coupling problems, it might help a small subclass of $f(R)$ theories constructed using the metric formulation, as we will now explain. The idea is to note that an extra degree of freedom may have an environmental contribution to the mass, which is natural in scalar-tensor theories, because the scalar field equation involves terms $\propto \tilde{T}$ as in Eq. (27). In dense environments the scalar would appear heavier and hence its long range forces can be Yukawa-suppressed. In particular, in [35] it was noted that there is a serendipitous numerical coincidence, that the density of water, in the units of the critical density of the universe, is close to the ratio of the Hubble length to a millimeter. Thus, if the theory yields an environmental scalar mass linearly proportional to the energy density of the environment, then the scalar could be suppressed in terrestrial conditions, and within the solar system, while simultaneously being as light as a quintessence field at horizon scales. The linear relationship between the mass and density is absolutely crucial. Otherwise the field which is massive enough on Earth to pass the bounds on the deviations from Newton’s law [36] will be too heavy at cosmological scales to serve as quintessence. Conversely, if it is quintessence-light at largest scales, it will violate these bounds [35]. On the other hand, unless the field is quintessence-light at horizon scales, it will not yield cosmic acceleration [35]. Thus for generic chameleon potentials which pass the bounds of [36], one must add dark energy by hand, in a way that does not relate it to the chameleon field⁶. An exception to this is the logarithmic potential $\tilde{V} = -\mu^4 \ln(\frac{\tilde{\chi}}{M})$ because it yields a mass exactly linearly proportional to $\rho_{\text{environment}}$ [35]. In this case the chameleon could be dark energy, and still remain phenomenologically viable. Since $f(R)$ models are all dual to such scalar-tensor theories, exploring the bounds like in [38] is at least reasonable, if one really wants to recast the chameleon quintessence as an $f(R)$ model. However, to have a chance to attribute cosmic acceleration to an $f(R)$ type action, as opposed to just adding a cosmological constant term on top of it, one needs to pick the functional form of f which reproduces the log potential of [35]. This is straightforward in principle, albeit mathematically contorted in practice. Indeed, the Legendre transformation rules, which yield $V(\Phi) = \varphi \partial_\varphi f - f$ and $\Phi = \partial_\varphi f$ combined with $\Phi = \frac{M_P^2}{2} e^{\sqrt{\frac{2}{3}} \frac{\tilde{\chi}}{M_P}}$ which follows from (20) for a canonically normalized scalar field of the standard formulation with

⁶We note that a different approach was recently pursued by Starobinsky [37], who proposed a class of $f(R)$ models with a ‘disappearing’ effective cosmological term, argued to arise from purely geometric considerations. In this model, there is an instability around flat space, and a channel for copious cosmological production of scalar modes [37].

$w = 0$ and the condition $\tilde{V} = e^{-2\sqrt{\frac{2}{3}}\hat{\chi}/M_P} V(\frac{M_P^2}{2} e^{\sqrt{\frac{2}{3}}\frac{\hat{\chi}}{M_P}})$ yield the differential equation for $f(\varphi)$,

$$-4\frac{\mu^4}{M_P^4}(\partial_\varphi f)^2 \ln\left(\sqrt{\frac{3}{2}}\frac{M_P}{M} \ln\left(\frac{2}{M_P^2}\partial_\varphi f\right)\right) = \varphi\partial_\varphi f - f. \quad (44)$$

Finding the solution of this equation in closed form looks rather intimidating. One might try to select suitable limits and seek the solutions as a series. Even so, one can see that the effective coupling parameter α of [35], is $\alpha = \frac{1}{2\sqrt{w+3/2}}$ (aside from an irrelevant sign convention). Therefore when $w = 0$ this is too large to support slow roll, which requires $\alpha < \frac{1}{4\sqrt{3}}$ [35]. However solving (44) really produces a one-parameter family of $f(R)$, and so one may yet carefully select the solution where a cosmological constant contribution and the effective light chameleon might combine to yield acceleration at the largest scales, while still modifying gravity and dark energy equation of state at shorter distances.

5 Summary

To conclude, we have clarified how to formulate scalar-tensor gravities, and models which reduce to them, in the first order formalism. To get correct Palatini gravities, one should use the Lagrange multiplier method, which preserves the canonical structure of the theory, and therefore yields the same scalar-tensor gravity as the standard metric formulation [30]. The apparent discrepancies between the naïve Palatini and the standard formulation based on Lagrange multipliers, encountered in [24]-[27], [17]-[23], arise because the naïve Palatini approach really replaces the theory for another, by changing the canonical momenta and therefore the dynamics, as compared to the standard approach. This explains why the same Lagrangian seemed to yield two different theories: in fact, they have different Hamiltonians, and so really are different from the very outset. The differences disappear only in the decoupling limit of the scalar sector, where the theories reduce to the ordinary General Relativity. In this case an accidental redundancy ensures that the naïve Palatini works there, as is familiar from the old lore. We have also investigated the decoupling limits and the strong coupling regimes of the theory. One of our side results was to establish that the so-called ‘Modified Source Gravity’ models suffer from strong coupling problems at very low scales, of order 10^{-3} eV, and hence cannot be a realistic approximation of our universe. Finally we observed that one might use chameleon mechanism to help decouple the extra scalar mode. However if one wants to relate the current cosmic acceleration to the modification of gravity as opposed to come from a completely separate dark energy sector, the chameleon mechanism may only work for a narrow class of theories, defined in our Eq. (44), even then requiring also a cosmological term. In sum, while seeking for modified gravity explanations of cosmic acceleration remains interesting, one needs to explore more dramatic ideas in order to go beyond the effective scalar-tensor models, which are well known and even better constrained.

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